Abstracts

Domains of discontinuity for Anosov representations

Daniele Alessandrini
(joint work with Sara Maloni, Nicolas Tholozan, Anna Wienhard)

Generalized flag manifolds. Let G be a connected semisimple Lie group with finite center. For example, G can be one of the classical matrix groups, such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}_0(p,q)$. We will consider the *generalized flag manifolds* of G: spaces of the form G/Q, where Q is a parabolic subgroup $Q < G$. For example, in the special case when $G = \mathrm{SL}(n,\mathbb{R})$, the spaces G/Q are the projective spaces, the Grassmannians, the partial flag manifolds and the full flag manifolds. These are classical geometric spaces endowed with a rich and interesting geometry.

Anosov representations. Let Γ be a torsion-free Gromov-hyperbolic group. We want to understand the geometric, topological and dynamical properties of an action of Γ on a generalized flag manifold G/Q. These actions correspond to representations

$$\rho : \Gamma \to G\,.$$

We will consider the set of all group homomorphisms of Γ in G, here denoted by

$$\mathrm{Hom}(\Gamma, G) = \{\ \rho : \Gamma \to G\ \}\,.$$

Among these representations, a special place is taken by the Anosov representations. They are the representations with the nicest dynamical properties. They are defined with reference to a parabolic subgroup $P < G$, where P may be different that the parabolic subgroup Q above. We will consider the set of P-Anosov representations, here denoted by

$$\mathrm{Anosov}_P(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)\,.$$

The P-Anosov representations of Γ are all discrete and faithful, and an important characteristic is that they admit a ρ-equivariant map

$$\xi : \partial_\infty \Gamma \to G/P\,.$$

These representations have the important property of being structurally stable, this means that a small deformation of a P-Anosov representation is still a P-Anosov representation. Equivalently, we can say that the set $\mathrm{Anosov}_P(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$. This property is useful for the construction of interesting examples of Anosov representations, see below.

Domains of discontinuity. The dynamics of the action of a P-Anosov representation ρ on a generalized flag manifold G/Q is described by the theory of domains of discontinuity, introduced by Guichard-Wienhard [3] and then generalized and improved by Kapovich-Leeb-Porti [4]. They give conditions for the existence of a cocompact domain of discontinuity for ρ in G/Q. They consider the space of relative positions of a point in G/P and a point in G/Q:

$$R = (G/P \times G/Q)/G\,.$$

This is a finite set with a rich combinatoric structure. Kapovich-Leeb-Porti define the notion of balanced ideal, a set $I \subset R$ with some special properties. This ideal represents the set of "bad" relative positions, that we don't want to have in the domain of discontinuity. Given $s \in G/P$, they define

$$K(s) = \{\ r \in G/Q \ \mid\ [(s,r)] \in I\ \} .$$

Recall that the P-Anosov representation admits the ρ-equivariant curve ξ as above. We want to remove the set

$$K_{\rho,I} = \bigcup_{t \in \partial_\infty \Gamma} K(\xi(t)) .$$

The complement of this set is the domain

$$\Omega_{\rho,I} = G/Q \smallsetminus K_{\rho,I} .$$

Kapovich-Leeb-Porti [4] proved that if I is a balanced ideal, the action of ρ on $\Omega_{\rho,I}$ is properly discontinuous, free and cocompact.

The quotient manifold. This construction gives us the closed manifold

$$M_{\rho,I} = \Omega_{\rho,I}/\rho .$$

Since this manifold is a quotient of a domain in G/Q, it carries a $(G, G/Q)$-structure, a geometric structure in the sense of Thurston. This gives us a way to construct manifolds $M_{\rho,I}$ with a large deformation space of $(G, G/Q)$-structures.

One limitation of this theory is that it doesn't say anything about the topology of the manifold $M_{\rho,I}$. Examples of such manifolds were studied by several authors, usually in the case when $\Gamma = \pi_1(S)$ is a surface group, see the references in [1]. From these examples we see that very often the manifold $M_{\rho,I}$ is a bundle over the surface. It would be tempting to conjecture that this is always true, but there are interesting counterexamples, given by Gromov-Lawson-Thurston [2], where $M_{\rho,I}$ cannot fiber over the surface. The examples in [2] are complicated, and suggest that it is impossible to give a general theorem that describes the topology of all the manifolds $M_{\rho,I}$.

Deformations of lattices. We will restrict to a special case, that is still very general and interesting, but is also more tractable. We will fix a connected semisimple Lie group H of real rank 1 with finite center, and we will assume that $\Gamma < H$ is a torsion-free uniform lattice in H. This class of groups Γ includes the surface groups and the fundamental groups of closed oriented (real, complex, quaternionic, octonionic) hyperbolic n-manifolds.

Given a representation $\iota : H \to G$, we can restrict it to Γ and obtain a representation $\rho_0 : \Gamma \to G$. These representations will be called *lattice representations* of Γ, and they are always P-Anosov with reference to some parabolic subgroup of G, see [3].

We can then use the property of structural stability, and deform the lattice representations a little bit. The deformed representation is still P-Anosov, and often it is Zariski-dense in G. We will say that a P-Anosov representation is a *deformation of a lattice representation* if it can be obtained by continuously deforming a lattice representation without leaving the space $\mathrm{Anosov_P}(\Gamma, G)$. In this way we can obtain many interesting examples of P-Anosov representations. Actually, most known Anosov representations are obtained from this construction, just because it is an easy way to construct them.

The topology of the quotient manifold. We will now present the main theorems in [1].

Theorem 1 (A., Maloni, Tholozan, Wienhard)**.** *Let $\Gamma < H$ be a torsion free uniform lattice. Recall that $\Gamma = \pi_1(T)$, where T is a closed oriented (real, complex, quaternionic, octonionic) hyperbolic manifold. Fix a representation $\iota : H \to G$, and restrict it to $\rho_0 : \Gamma \to G$. Fix a $P < G$ such that ρ_0 is P-Anosov. Let $\rho : \Gamma \to G$ be a deformation of the representation ρ_0. Choose a parabolic subgroup Q such that there exists a balanced ideal I of relative positions between G/P and G/Q, and let $M = M_{\rho,I}$ be the quotient $\Omega_{\rho,I}/\rho$.*

Then M is a smooth fiber bundle over the manifold T.

We can describe the structure group of this bundle and characterize the bundle.

Theorem 2 (A., Maloni, Tholozan, Wienhard)**.** *With the same notation as in the previous theorem, let F denote the fiber of the bundle:*

$$F \to M \to T\,.$$

Denote by K the maximal compact subgroup of H, and by $S_H = H/K$ the symmetric space of H, a (real, complex, quaternionic, octonionic) hyperbolic space. Consider the principal K-bundle

$$K \to H/\Gamma \to S_H/\Gamma = T\,.$$

Then F has a K-action, the bundle $M \to T$ has structure group K given by this action, and the bundle $M \to T$ is associated to the principal bundle $H/\Gamma \to T$.

Theorem 1 and 2 fully describe the topology of M, except for the fiber F. We know the topology of the fiber F in many special cases (see [1] and the references therein), but a general description of the fiber F is still not known.

References

[1] D. Alessandrini, S. Maloni, N. Tholozan, A. Wienhard, *Fiber bundles associated with Anosov representations*, preprint (2023) `arXiv:2303.10786`.

[2] M. Gromov, H. B. Lawson, W. Thurston, *Hyperbolic 4-manifolds and conformally flat 3-manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 27-45.

[3] O. Guichard, A. Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), no. 2, 357-438.

[4] M. Kapovich, B. Leeb, J. Porti, *Dynamics on flag manifolds: domain of proper discontinuity and cocompactness*, Geom. Topol. **22** (2018), no. 1, 157-234.